

Spacetime-noncommutativity regime of Loop Quantum Gravity

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A recent study by Bojowald and Paily [1] provided a path toward the identification of an effective quantum-spacetime picture of Loop Quantum Gravity, applicable in the “Minkowski regime”, the regime where the large-scale (coarse-grained) spacetime metric is flat. A pivotal role in the analysis is played by Loop-Quantum-Gravity-based modifications to the hypersurface deformation algebra, which leave a trace in the Minkowski regime. We here show that the symmetry-algebra results reported by Bojowald and Paily are consistent with a description of spacetime in the Minkowski regime given in terms of the κ -Minkowski noncommutative spacetime, whose relevance for the study of the quantum-gravity problem had already been proposed for independent reasons.

I. INTRODUCTION

Over the last decade the quantum-gravity literature has been increasingly polarizing into a top-down approach and a bottom-up approach. The top-down approach attempts to provide models that could potentially solve at once all aspects of the quantum-gravity problem, but typically involves formalisms of very high complexity, rather unmanageable for obtaining physical intuition about observable (and potentially testable) features. The bottom-up approach relies on relatively simpler models, suitable for describing only a small subset of the departures from standard physics that the quantum-gravity realm is expected to host, but has the advantage of producing better opportunities for experimental testing [2]. A good synergy between the two approaches would be desirable: from the top we could obtain guidance on which are the most significant structures to be taken into account in more humble formalizations, and from the bottom we could develop insight on how to handle those structures, hopefully also in terms of experimental tests. Unfortunately very little has been accomplished so far in the spirit of such a synergy, but we here offer a contribution toward establishing a link between Loop Quantum Gravity (LQG) [3–5], one of the most studied approaches aiming for a full solution of the quantum-gravity problem, and approaches focusing on the assumption of non-commutativity of coordinates in the Minkowski regime of quantum gravity [6–9].

Both the LQG approach and the spacetime-noncommutativity approach involve the possibility that the geometry of spacetime might be quantized in the quantum-gravity realm. In LQG one in principle obtains a very powerful picture of this quantum geometry, applicable to all regimes of quantum gravity, but the complexity of the formalism is such that *de facto* there is no physical regime of quantum gravity for which we are presently able to use LQG for an intuitive (intelligible) characterization of the novel physical properties that would result from the quantum-geometric properties. Spacetime non-commutativity takes the more humble approach of postulating one or another form of noncommutativity of space-

time coordinates, hoping that it might be applicable in the Minkowski regime, but has the advantage of leading to several rather intuitive findings about the physical implications of these assumptions, some of which attracted even some interest in phenomenology [2, 10–13]. A link between LQG and spacetime noncommutativity is solidly established for dimensionally-reduced 3D quantum gravity [14–17], but it remains so far unclear whether a generalization of those results is applicable to the 4D case of real physical interest.

We here take what might be a significant step toward establishing a link between (4D) LQG and spacetime noncommutativity. The starting point for our analysis is provided by a recent publication by Bojowald and Paily [1], which contemplated some LQG-based modifications to the hypersurface deformation algebra. Interestingly, Bojowald and Paily found that these modifications of the hypersurface deformation algebra leave a trace in the Minkowski regime, characterized by a suitable modification of the Poincaré algebra, and correctly concluded that a quantum-spacetime dual to that deformed Poincaré algebra should then give the quantum-geometry description of LQG in the Minkowski regime. Indeed deformed Poincaré algebras are characteristic of the structure of DSR-relativistic theories, first introduced in Ref. [18] (also see the follow-up studies in Refs. [8, 19–21]), and for such theories one expects in general that the duality between Minkowski spacetime and the classical Poincaré algebra be preserved in the form of a duality between a suitably deformed Poincaré algebra and a “quantum Minkowski spacetime”. Bojowald and Paily also contemplated the possibility that this quantum-spacetime picture be given in terms of the much-studied “ κ -Minkowski noncommutative spacetime” [22, 23], but explored this possibility only preliminarily by relying on a specific *ansatz* for the representations of the action of the relevant deformed Poincaré algebra on κ -Minkowski coordinates. This preliminary exploration gave negative results (incompatibility between the *ansatz* for the representations and some properties of κ -Minkowski coordinates), leading Bojowald and Paily to tentatively conclude that κ -Minkowski might not provide the needed

quantum-spacetime picture.

We here expose some limitations of the *ansatz* adopted by Bojowald and Paily. By considering a more general class of possible representations of the action of the relevant deformed Poincaré algebra on κ -Minkowski coordinates we establish the compatibility between κ -Minkowski and the findings reported by Bojowald and Paily for the LQG-deformed hypersurface deformation algebra. Our analysis also leads to the identification of the “coproduct structure” which is to be expected in regimes characterized by κ -Minkowski noncommutativity, relevant for the description of the action of relativistic-symmetry transformations on products of fields. Finding evidence of this coproduct structure on the LQG side would provide final proof of the correspondence between LQG and κ -Minkowski.

II. DEFORMED HYPERSURFACE-DEFORMATION ALGEBRA

The Hamiltonian formulation of general relativity allows to encode the general covariance of the theory in the algebra closed by the scalar ($H[N]$) and vector ($D[N^i]$) constraints, the so-called Hypersurface-Deformation Algebra (HDA) [24–26]:

$$\begin{aligned} \{D[M^k], D[N^j]\} &= D[\mathcal{L}_{\vec{M}} N^k], \\ \{D[N^k], H[M]\} &= H[\mathcal{L}_{\vec{N}} M], \\ \{H[N], H[M]\} &= D[h^{jk}(N\partial_j M - M\partial_j N)], \end{aligned} \quad (1)$$

where $H[N]$ and $D[N^i]$ depend respectively on the lapse N and shift N^i functions [25].

It is natural to ask if the HDA should be deformed in quantum gravity due to the presence of quantum-geometry corrections, such as those arising in LQG [27–30]. Indeed, recently there has been a growing effort in studying quantum deformations of Eqs. (1), especially in the context of models motivated by LQG [29–32]. For our purposes here, it is of particular interest the analysis reported in Ref. [1], which shows that a particular case of deformed HDA reduces to a Planck-scale-deformed Poincaré algebra if one takes the flat-spacetime limit. Most notably the relevant deformations of the Poincaré algebra are qualitatively of the type known to arise in the description of the relativistic symmetries of noncommutative spacetimes.

From a wider perspective this shows a possible path from ambitious quantum-gravity theories to certain phenomenological opportunities available [2] in the “Minkowski regime”: within the full quantum-gravity theory one should be in position to study the deformations of the HDA; then by taking the Minkowski-regime limit of such a deformed HDA one should find the corresponding deformation of the Poincaré algebra applicable to the Minkowski limit of the model, and finally one could obtain an effective quantum-Minkowski-spacetime picture, by duality with the relevant deformed Poincaré

algebra. The analysis reported by Bojowald and Paily [1] provides an opportunity for a first application of this strategy of analysis, relevant for the LQG approach. The first two steps have already been accomplished in Ref. [1], by motivating a specific deformation of the HDA and by finding the associated deformation of the Poincaré algebra applicable to the Minkowski limit. For the third step, the one providing an effective quantum-Minkowski-spacetime picture, the analysis reported in Ref. [1] was inconclusive for reasons already discussed in the previous section. Since our objective here is to take this important third step, we find appropriate to summarize briefly in this section some key aspects of the LQG-deformed HDA, from the perspective adopted in Ref. [1].

As mentioned, it is expected that Eqs. (1) should receive quantum-gravity corrections [27, 31, 33, 34], and in particular this is expected for the LQG scenario [27, 29, 32, 35]. A fully deductive derivation of the deformed HDA within LQG is at present beyond our technical abilities, so different authors have relied on different approximation schemes, but all results agree on the following form for the deformed HDA [29, 31, 32, 36]:

$$\begin{aligned} \{D[M^a], D[N^a]\} &= D[\mathcal{L}_{\vec{M}} N^a], \\ \{D[N^a], H^Q[M]\} &= H^Q[\mathcal{L}_{\vec{N}} M], \\ \{H^Q[M], H^Q[N]\} &= D[\beta h^{ab}(M\partial_b N - N\partial_b M)], \end{aligned} \quad (2)$$

where $H^Q[N]$ denotes a deformed (“quantum”) scalar constraint and β depends on the specific corrections that are taken into account. A key challenge is to find an appropriate representation of constraints as operators on a Hilbert space, and so far no proposal has fully accomplished this task. However, several techniques have been developed and some promising candidates for the quantum Hamiltonian operator ($H^Q[N]$) have been proposed [5, 28, 35, 37, 38]. In particular, in spherically-symmetric models [29, 30], which are here of interest, some of the quantum corrections, namely the local (i.e. point-like) holonomy corrections [29, 30, 37], have been successfully implemented, and the corresponding quantized version of the scalar constraint can still close an algebra provided that it is properly deformed as in Eqs. (2).

For spherically-symmetric analyses it is useful to write the 3-metric h_{ij} in suitably adapted fashion, and for this purpose it is useful to describe the densitized triads [5, 39] as follows:

$$\begin{aligned} E &= E_i^a \tau^i \frac{\partial}{\partial x^a} = E^r(r) \tau_3 \sin \theta \frac{\partial}{\partial r} + \\ &+ E^\varphi(r) \tau_1 \sin \theta \frac{\partial}{\partial \theta} + E^\varphi(r) \tau_2 \frac{\partial}{\partial \varphi}, \end{aligned} \quad (3)$$

where $\tau_j = -\frac{1}{2}i\sigma_j$ represent $SU(2)$ generators. The densitized triads are canonically conjugate to the extrinsic curvature components, which, in presence of spherical symmetry, are conveniently described as follows [29, 30]:

$$\begin{aligned} K &= K_a^i \tau_i dx^a = K_r(r) \tau_3 dr + K_\varphi(r) \tau_1 d\theta + \\ &+ K_\varphi(r) \tau_2 \sin \theta d\varphi. \end{aligned} \quad (4)$$

It has been shown [29, 30, 37] that the deformation function β depends on the angular component of the extrinsic curvature K_φ as follows:

$$\beta = \cos(2\delta K_\varphi),$$

where δ is a parameter which can be related to the square root of the minimum eigenvalue of the area operator [35, 37]. Since we would like to obtain, in the appropriate limit, a deformed Poincaré algebra, it is convenient to write β in terms of symmetry generators, and for this purposes it is valuable to observe that observables of the Brown York momentum [40],

$$P = 2 \int_{\partial\Sigma} d^2z v_b (n_a \pi^{ab} - \bar{n}_a \bar{\pi}^{ab}), \quad (5)$$

can be identified by extrinsic curvature components provided a suitable choice for $\delta \propto |E^r|^{-\frac{1}{2}}$. In Eq.(5), we have that $v_a = \partial/\partial x^a$, n_a is the conormal of the boundary of the spatial region Σ , and π^{ab} plays the role of the gravitational momentum. From this, it is possible to establish that the radial Brown-York momentum P_r is related to the extrinsic curvature component K_φ in the following way [1]:

$$P_r = -\frac{K_\varphi}{\sqrt{|E^r|}}. \quad (6)$$

In order to obtain the Poincaré algebra from the HDA one should consider flat spatial slices of spacetime, *i.e.* a Euclidean three metric $h_{ij} = \delta_{ij}$, and also take a combination [41] of the Killing vectors of Minkowski spacetime as lapse and shift functions:

$$\begin{aligned} N &= \Delta t + v_k x^k \\ N^i &= \Delta x^i + \phi^j \epsilon^{ijk} x^k \end{aligned} \quad (7)$$

It is generally expected (see Ref. [1, 42] and references therein) that this very direct connection between Poincaré algebra and HDA should still be present when quantum-gravity effects are taken into account, and therefore if the HDA is affected by quantum-gravity modifications, then also the Poincaré algebra should be correspondingly modified. We then turn to the deformed HDA of Eq.(2), and notice that, in presence of spherical symmetry, and taking into account Eq.(6), the deformation function β is a function of the generator of spatial translations, *i.e.* $\beta = \cos(\lambda P_r)$, where λ is a parameter of the order of the Planck length. The net result is that, as a result of Eq. (2), the relevant Poincaré algebra is characterized by a deformed commutator between boost generator and generator of time translations:

$$[B_r, P_0] = i P_r \cos(\lambda P_r). \quad (8)$$

Since only the Poisson bracket involving two scalar constraints is quantum corrected (see Eqs. (2)), the other commutators are undeformed, *i.e.* $[B_r, P_r] = i P_0$ and $[P_0, P_r] = 0$.

III. COMPATIBILITY WITH κ -MINKOWSKI

Our next task is to show that the operators B_r, P_r and P_0 generate the deformed-Poincaré-symmetry transformations which are symmetries of the κ -Minkowski noncommutative spacetime. What is here relevant is the spherically-symmetric version of the κ -Minkowski noncommutativity of spacetime coordinates [22, 23]:

$$[X_0, X_r] = i\lambda X_r. \quad (9)$$

As mentioned above, already Bojowald and Paily had contemplated the possibility [1] that B_r, P_r and P_0 might generate the deformed-Poincaré-symmetry transformations which are symmetries of the κ -Minkowski noncommutative spacetime. They however somehow assumed the following representations:

$$\begin{aligned} B_r &= x_r p_0 - \cos(\lambda p_r) x_0 p_r \\ P_r &= p_r \\ P_0 &= p_0, \end{aligned} \quad (10)$$

in terms of standard operators such that $[x_r, p_r] = i$, $[x_0, p_0] = -i$, $[x_r, p_0] = 0$, $[x_0, p_r] = 0$, $[x_0, x_r] = 0$, $[p_0, p_r] = 0$, and they correctly found (also relying on results which had been previously reported in Ref.[43]) that this representation is incompatible with the structure of κ -Minkowski spacetime.

We here notice that the representation (10) adopted by Bojowald and Paily is only one of many possibilities that can be tried. Experience working with spacetime noncommutativity teaches that it is actually rather hard to guess correctly such representations. A constructive approach is usually appropriate, and we therefore seek a suitable representation of B_r, P_r and P_0 within a rather general *ansatz*:

$$\begin{aligned} B_r &= F(p_0, p_r) X_r p_0 - G(p_0, p_r) X_0 p_r, \\ P_r &= Z(p_r), \\ P_0 &= p_0, \end{aligned} \quad (11)$$

where $F(p_0, p_r)$, $G(p_0, p_r)$ and $Z(p_r)$ are functions of the translation generators to be determined by enforcing compatibility with the deformed algebra (8). Our *ansatz* involves the κ -Minkowski noncommutative coordinates X_r, X_0 , but this is only for convenience: it is well known (see, *e.g.*, Ref.[43]) that one can represent the coordinates X_r, X_0 in terms of the x_r, x_0, p_r used by Bojowald and Paily (with $X_r = x_r$ and $X_0 = x_0 - \lambda x_r p_r$), so the class of representations covered by our *ansatz* is qualitatively of the same type as the one of the representation considered by Bojowald and Paily. The real difference resides in the structure of the representations and the fact that we will seek a suitable representation by determining the functions $F(p_0, p_r)$, $G(p_0, p_r)$ and $Z(p_r)$, rather than try to guess.

Explicitly our objective is to find choices of $F(p_0, p_r)$, $G(p_0, p_r)$ and $Z(p_r)$ such that (8) is satisfied, with

$[B_r, P_r] = iP_0$ and $[P_0, P_r] = 0$. We came to notice that this is assured if $Z(p_r)$ is a solution of the equation

$$\lambda Z(p_r) \sin(\lambda Z(p_r)) + \cos(\lambda Z(p_r)) = \frac{\lambda^2 p_r^2}{2} + 1. \quad (12)$$

and then $F(p_0, p_r)$ and $G(p_0, p_r)$ are given in terms of such a solution for $Z(p_r)$ through the following equations:

$$G(p_r) = \frac{Z(p_r) \cos(\lambda Z(p_r))}{p_r}, \quad (13)$$

$$F(p_0, p_r) = G(p_r) e^{\lambda p_0} = \frac{Z(p_r) \cos(\lambda Z(p_r)) e^{\lambda p_0}}{p_r},$$

So we have reduced the problem of finding representations on κ -Minkowski of the Bojowald-Paily deformed Poincaré algebra to the problem of finding solutions to equation (12). Of course we must also enforce that such solutions $Z(p_r)$ satisfy the limiting condition $\lim_{\lambda \rightarrow 0} \frac{Z(p_r)}{p_r} = 1$, since the undeformed representation of Poincaré generators must be recovered when the non-commutativity is turned off.

We were unable to find an explicit all-order expression for such a solution $Z(p_r)$, but we find that its perturbative derivation (as a series of powers of λ) is always possible and straightforward up to the desired perturbative order. In particular, to quartic order in the parameter λ the needed solution $Z(p_r)$ takes the form:

$$Z(p_r) = p_r + \frac{1}{8} \lambda^2 p_r^3 + \frac{55}{1152} \lambda^4 p_r^5. \quad (14)$$

Notice that on the basis of remarks given above evidently p_r acts on κ -Minkowski noncommutative coordinates as follows:

$$[p_r, X_0] = i\lambda p_r, \quad [p_r, X_r] = -i \quad (15)$$

while for what concerns p_0 one has

$$[p_0, X_0] = i, \quad [p_0, X_r] = 0 \quad (16)$$

Equipped with this final specification one can easily check explicitly that (as ensured automatically by our constructive procedure) the representation here obtained up to quartic order in λ for the generators B_r, P_r and P_0 satisfies all the Jacobi identities involving these generators and κ -Minkowski coordinates. For example one has that:

$$\begin{aligned} & [[B_r, X_r], X_0] + [[X_0, B_r], X_r] + [[X_r, X_0], B_r] = \\ & -i \left[\frac{(Z' \cos(\lambda Z) - \lambda Z Z' \sin(\lambda Z)) p_r - Z \cos(\lambda Z)}{p_r^2} x_r p_0, X_0 \right] + \\ & + i \left[\frac{(Z' \cos(\lambda Z) - \lambda Z Z' \sin(\lambda Z)) p_r - Z \cos(\lambda Z)}{p_r^2} x_0 p_r, X_0 \right] + \\ & - \left[i \frac{Z \cos(\lambda Z)}{p_r} x_r - \lambda [B_r, X_r] p_r - i \lambda \frac{Z \cos(\lambda Z)}{p_r} x_r p_0, X_r \right] + \\ & + \left[i \frac{Z \cos(\lambda Z)}{p_r} x_0, X_0 \right] + i \lambda [B_r, X_r] = 0 \end{aligned}$$

where $Z' = \frac{dZ(p_r)}{dp_r}$.

IV. COPRODUCTS AND CHOICE OF “BASIS”

The results we reported in the previous section provide strong encouragement for the possibility that the quantum-Minkowski spacetime emerging from the Bojowald-Paily analysis is the κ -Minkowski noncommutative spacetime. Our next objective is to discuss some implications of these findings.

The first point we make is that in order to have a description of the symmetries of a noncommutative spacetime one should specify not only the commutator of symmetry generators but also their coproducts [22, 23]. From the representations of B_r, P_r and P_0 we derived in the previous section one easily finds (with standard steps of derivation which have been discussed in several publications such as Refs.[22, 23, 44]) that these coproducts are given by:

$$\begin{aligned} \Delta B_r &= B_r \otimes 1 + 1 \otimes B_r - \lambda P_0 \otimes B_r + \frac{1}{8} \lambda^2 P_r^2 \otimes B_r \\ &+ \frac{1}{2} \lambda^2 P_0^2 \otimes B_r - \frac{3}{8} \lambda^2 B_r \otimes P_r^2 - \frac{3}{4} \lambda^2 P_r B_r \otimes P_r \\ &- \frac{3}{4} \lambda^2 P_r \otimes P_r B_r - \frac{5}{8} \lambda^3 P_0 P_r^2 \otimes B_r + \frac{3}{4} \lambda^3 P_0 P_r \otimes P_r B_r \\ &- \frac{3}{4} \lambda^3 P_r^2 B_r \otimes P_0 - \frac{3}{4} \lambda^3 P_r^2 \otimes P_0 B_r - \frac{3}{4} \lambda^3 P_r B_r \otimes P_0 P_r \\ &- \frac{3}{4} \lambda^3 P_r \otimes P_0 P_r B_r + \frac{67}{1152} \lambda^4 P_r^4 \otimes B_r + \frac{15}{64} \lambda^4 P_r^2 \otimes P_r^2 B_r \\ &- \frac{1}{8} \lambda^4 P_0^4 \otimes B_r + \frac{9}{16} \lambda^4 P_0^2 P_r^2 \otimes B_r + \frac{15}{64} \lambda^4 P_r^2 B_r \otimes P_r^2 \\ &- \frac{167}{288} \lambda^4 P_r^3 B_r \otimes P_r - \frac{59}{288} \lambda^4 P_r \otimes P_r^3 B_r \\ &- \frac{97}{144} \lambda^4 P_r^3 \otimes P_r B_r - \frac{3}{8} \lambda^4 P_0^2 P_r \otimes P_r B_r \\ &+ \frac{3}{4} \lambda^4 P_0 P_r^2 \otimes P_0 B_r + \frac{3}{4} \lambda^4 P_0 P_r \otimes P_0 P_r B_r \\ &+ \frac{11}{144} \lambda^4 P_r B_r \otimes P_r^3 - \frac{3}{4} \lambda^4 P_r^2 B_r \otimes P_0^2 \\ &- \frac{3}{4} \lambda^4 P_r^2 \otimes P_0^2 B_r - \frac{5}{1152} \lambda^4 B_r \otimes P_r^4 \\ &- \frac{3}{8} \lambda^4 P_r B_r \otimes P_0^2 P_r - \frac{3}{8} \lambda^4 P_r \otimes P_0^2 P_r B_r \end{aligned}$$

$$\begin{aligned} \Delta P_r &= P_r \otimes 1 + 1 \otimes P_r + \lambda P_r \otimes P_0 + \frac{1}{2} \lambda^2 P_r \otimes P_0^2 \\ &- \frac{1}{8} \lambda^2 P_r \otimes P_r^2 + \frac{3}{8} \lambda^2 P_r^2 \otimes P_r + \frac{1}{4} \lambda^3 P_r^3 \otimes P_0 \\ &+ \frac{3}{8} \lambda^3 P_r \otimes P_0 P_r^2 + \frac{3}{4} \lambda^3 P_r^2 \otimes P_0 P_r + \frac{1}{2} \lambda^4 P_r^3 \otimes P_0^2 \\ &- \frac{49}{1152} \lambda^4 P_r \otimes P_r^4 + \frac{11}{36} \lambda^4 P_r^3 \otimes P_r^2 - \frac{1}{8} \lambda^4 P_r \otimes P_0^4 \\ &+ \frac{7}{16} \lambda^4 P_r \otimes P_0^2 P_r^2 + \frac{1}{18} \lambda^4 P_r^2 \otimes P_r^3 \\ &+ \frac{167}{1152} \lambda^4 P_r^4 \otimes P_r + \frac{3}{4} \lambda^4 P_r^2 \otimes P_0^2 P_r \end{aligned}$$

$$\begin{aligned}
\Delta P_0 &= P_0 \otimes 1 + 1 \otimes P_0 + \lambda P_r \otimes P_r + \frac{1}{2} \lambda^2 P_0 \otimes P_0^2 \\
&+ \frac{1}{2} \lambda^2 P_0^2 \otimes P_0 - \frac{1}{2} \lambda^2 P_0 \otimes P_r^2 - \lambda^2 P_0 P_r \otimes P_r \\
&+ \frac{1}{2} \lambda^2 P_r^2 \otimes P_0 - \frac{1}{8} \lambda^3 P_r \otimes P_r^3 + \frac{3}{8} \lambda^3 P_r^3 \otimes P_r \\
&+ \frac{1}{2} \lambda^3 P_0^2 P_r \otimes P_r - \lambda^3 P_0 P_r^2 \otimes P_0 + \frac{1}{4} \lambda^4 P_r^4 \otimes P_0 \\
&- \frac{1}{8} \lambda^4 P_0 \otimes P_0^4 - \frac{1}{8} \lambda^4 P_0^4 \otimes P_0 + \frac{1}{8} \lambda^4 P_0 P_r \otimes P_r^3 \\
&+ \frac{1}{4} \lambda^4 P_0 \otimes P_0^2 P_r^2 + \frac{3}{4} \lambda^4 P_0^2 P_r^2 \otimes P_0 - \frac{7}{8} \lambda^4 P_0 P_r^3 \otimes P_r
\end{aligned}$$

where again we are working to quartic order in λ . Re-assuringly the coproducts “close”, *i.e.* they can be expressed in terms of the generators B_r, P_r and P_0 , which is considered as a key consistency criterion [22, 23, 44] for the description of the symmetries of a noncommutative spacetime. We shall offer some comments here below on the awkwardly lengthy expressions these coproducts have. Before we get to that we add one more observation concerning the fact that the results we are reporting must fall within the structure of the κ -Poincaré Hopf algebra. Indeed, it has been independently established [6, 22, 23, 44] that the κ -Poincaré Hopf algebra describes the symmetries of the κ -Minkowski noncommutative spacetime. The κ -Poincaré Hopf algebra can present itself, as far as explicit formulas are concerned, in some rather different ways, depending on the conventions adopted. This is because for a Hopf algebra not only linear but also non-linear redefinitions of the generators provide admissible “bases”. This is why it is often not easy to recognize, as in the case here of interest, that a given set of commutation relations is actually a basis of the κ -Poincaré Hopf algebra. However, having established above that the Bojowald-Paily operators B_r, P_r and P_0 describe the relativistic symmetries of the κ -Minkowski spacetime, we must now infer that B_r, P_r and P_0 must give a basis of κ -Poincaré. The most direct way for showing that a given set of commutation rules is a basis for κ -Poincaré is to show that there is nonlinear redefinition of the generators which maps them into a known basis of κ -Poincaré. In order to accomplish this task we took as reference the most used basis of κ -Poincaré, the so-called bicrossproduct basis, characterized by the following commutation relations [22, 23]:

$$\begin{aligned}
[\mathcal{P}_0, \mathcal{P}_r] &= 0, \quad [B_r, \mathcal{P}_0] = i P_r, \\
[B_r, \mathcal{P}_r] &= i \frac{1 - e^{-2\lambda \mathcal{P}_0}}{2\lambda} - i \frac{\lambda}{2} \mathcal{P}_r^2,
\end{aligned} \tag{17}$$

where we introduced the notation $\mathcal{P}_0, \mathcal{P}_r, B_r$ for the generators of the bicrossproduct basis.

We have obtained the relationship between the Bojowald-Paily operators B_r, P_r and P_0 and bicrossproduct-basis generators $\mathcal{P}_0, \mathcal{P}_r, B_r$ in terms of the function $Z(p_r)$ which, as shown in the previous

section, must solve Eq.(12) in order for us to have a consistent representation of B_r, P_r and P_0 on the κ -Minkowski spacetime. This relationship takes the form:

$$\begin{aligned}
B_r &= \frac{Z(\mathcal{P}_r e^{\lambda \mathcal{P}_0}) \cos(\lambda Z(\mathcal{P}_r e^{\lambda \mathcal{P}_0}))}{\mathcal{P}_r e^{\lambda \mathcal{P}_0}} B_r, \\
P_r &= Z(\mathcal{P}_r e^{\lambda \mathcal{P}_0}), \\
P_0 &= \frac{\sinh(\lambda \mathcal{P}_0)}{\lambda} + \frac{\lambda}{2} \mathcal{P}_r^2 e^{\lambda \mathcal{P}_0},
\end{aligned} \tag{18}$$

which (since we have an explicit result for $Z(p_r)$ to quartic order in λ) we can render explicit to quartic order in λ :

$$\begin{aligned}
B_r &= (1 + \lambda^2 \mathcal{P}_0^2 - \frac{3}{8} \lambda^2 \mathcal{P}_r^2 - \frac{3}{4} \lambda^3 \mathcal{P}_0 \mathcal{P}_r^2 + \\
&- \frac{5}{4} \lambda^4 \mathcal{P}_r^2 \mathcal{P}_0^2 - \frac{113}{1152} \lambda^4 \mathcal{P}_r^4) B_r
\end{aligned} \tag{19}$$

$$\begin{aligned}
P_r &= \mathcal{P}_r + \lambda \mathcal{P}_r \mathcal{P}_0 + \frac{\lambda^2}{2} \mathcal{P}_r \mathcal{P}_0^2 + \frac{\lambda^2}{8} \mathcal{P}_r^3 + \frac{3}{8} \lambda^3 \mathcal{P}_0 \mathcal{P}_r^3 + \\
&+ \frac{\lambda^3}{6} \mathcal{P}_r \mathcal{P}_0^3 + \frac{9}{16} \lambda^4 \mathcal{P}_0^2 \mathcal{P}_r^3 + \frac{\lambda^4}{24} \mathcal{P}_r \mathcal{P}_0^4 + \frac{55}{1152} \lambda^4 \mathcal{P}_r^5
\end{aligned} \tag{20}$$

$$\begin{aligned}
P_0 &= \mathcal{P}_0 + \frac{\lambda}{2} \mathcal{P}_r^2 + \frac{\lambda^2}{6} \mathcal{P}_0^3 + \frac{\lambda^2}{2} \mathcal{P}_0 \mathcal{P}_r^2 + \\
&+ \frac{\lambda^3}{4} \mathcal{P}_0^2 \mathcal{P}_r^2 + \frac{\lambda^4}{120} \mathcal{P}_0^5 + \frac{\lambda^4}{12} \mathcal{P}_0^3 \mathcal{P}_r^2
\end{aligned} \tag{21}$$

The last point we want to make in this section is connected with this issue of the choice of basis for κ -Poincaré, and it will take us back to the awkwardly lengthy formulas we encountered in the description of coproducts. Different bases of κ -Poincaré provide an equally acceptable mathematical characterization of the symmetries; however, it is known that some bases provide a more intuitive description of the associated physical properties. The main issue here concerns the relationship between the properties of the translation generators and the properties of the energy-momentum charges: this relationship is rather intuitive in some bases but potentially misleading in some other basis. Here relevant is the fact that non-linear redefinitions of generators give rise to equivalent mathematical pictures, while non-linear redefinitions of the energy-momentum charges are physically significant. In the context of the analysis we are here reporting these concerns take shape by looking at the form of the deformed mass Casimir obtained for the P_0, P_r, B_r basis of κ -Poincaré. From Eq. (8), with $[B_r, P_r] = i P_0$ and $[P_0, P_r] = 0$, it follows that this deformed mass Casimir takes the form:

$$P_0^2 = \frac{2}{\lambda^2} (\lambda P_r \sin(\lambda P_r) + \cos(\lambda P_r) - 1). \tag{22}$$

Usually one is able to obtain from the form of the mass Casimir the form of the on-shell relation by simply replacing the translation generators with the energy-momentum charges, but in this case we conjecture that

this might be inappropriate. This conjecture comes from observing that interpreting (22) as an on-shell relation for the energy-momentum charges one is led to the puzzling picture shown in Figure 1, with the energy which is not a monotonic function of spatial momentum (and at large spatial momentum one has that the energy decreases as spatial momentum increases). We would argue that the implications of Figure 1 are just as awkward as the length formulas needed to describe results. We conjecture that eventually in the study of this scenario a different basis will be motivated, leading to a monotonic on-shellness relation and simpler formulas for coproducts. It would of course be particularly significant if the LQG side of this scenario provided some input on the correct choice of basis.

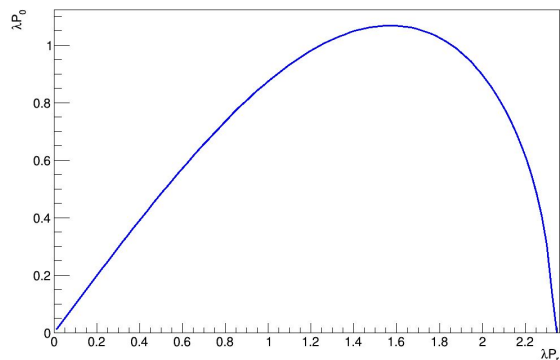


FIG. 1. Behavior (for $0 \leq \lambda P_1 < 2.35$) of an on-shell relation inspired by the mass Casimir of Eq. (22).

V. OUTLOOK

We believe the analysis here reported might be a significant step toward the description of the Minkowski limit of LQG. From a broader perspective we are seeing the type of back-and-forth steps that in general will be required in order to establish a connection between top-down approaches and bottom-up ones. The anal-

ysis reported in Ref.[1], taking LQG as starting point, produced some commutation relations for the Minkowski limit which we could here analyze from the viewpoint of κ -Minkowski noncommutativity. In turn, our analysis led us to establish a specific form for the coproducts, which, as stressed above, for consistency should be found to play a role in the action of relativistic-symmetry transformations on the product of states within the LQG formalism. We hope the challenge of seeking such a role for our coproduct is taken by LQG experts, as it might lead to striking developments.

Similarly, through the connection with the κ -Minkowski spacetime it was natural for us to conjecture that the translation generators adopted in Ref.[1] might not be the most natural choice, at least not in the sense that their properties reflect those of the energy-momentum charges. This was suggested by the awkwardness of the implied on-shell relation and by the cumbersome form that the coproducts take when written in terms of those translation generators.

Even looking beyond the contexts of LQG research and κ -Minkowski research, we have strong expectations for the usefulness of the strategy of analysis introduced in Ref.[1], and here further developed. This strategy essentially sees the (modifications of the) hypersurface deformation algebra as the point of connection between the top-down and bottom-up approaches. For a top-down approach obtaining results for the modifications of the hypersurface deformation algebra should be viewed as a very natural goal, and then, as shown here and in Ref.[1], the path from the hypersurface deformation algebra to a quantum-spacetime description of the Minkowski limit should be manageable. It would be particularly interesting to see this strategy implemented in other top-down approaches besides LQG.

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